International Mathematics Competition for University Students
July 25–30 2009, Budapest, Hungary
Day 2

Problem 1.
Let ℓ be a line and P a point in $\mathbb{R}^3$. Let S be the set of points X such that the distance from X to ℓ is greater than or equal to two times the distance between X and P. If the distance from P to ℓ is $d > 0$, find the volume of S.

Problem 2.
Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a two times differentiable function satisfying $f(0) = 1$, $f'(0) = 0$, and for all $x \in [0, \infty)$

$$f''(x) - 5f'(x) + 6f(x) \geq 0.$$

Prove that for all $x \in [0, \infty)$

$$f(x) \geq 3e^{2x} - 2e^{3x}.$$

Problem 3.
Let A and B be two complex square matrices such that

$$A^2B + BA^2 = 2ABA.$$

Prove that there exists a positive integer k such that $(AB - BA)^k = 0$.

Problem 4.
Let p be a prime number and $\mathbb{F}_p$ be the field of residues modulo p. Let W be the smallest set of polynomials with coefficients in $\mathbb{F}_p$ such that

- the polynomials $x + 1$ and $x^{p-2} + x^{p-3} + \cdots + x^2 + 2x + 1$ are in W, and
- for any polynomials $h_1(x)$ and $h_2(x)$ in W the polynomial $r(x)$, which is the remainder of $h_1(h_2(x))$ modulo $x^p - x$, is also in W.

How many polynomials are there in W?

Problem 5.
Let $\mathbb{M}$ be the vector space of $m \times p$ real matrices. For a vector subspace $S \subseteq \mathbb{M}$, denote by $\delta(S)$ the dimension of the vector space generated by all columns of all matrices in S.

Say that a vector subspace $T \subseteq \mathbb{M}$ is a *covering matrix space* if

$$\bigcup_{A \in T,\, A \neq 0} \ker A = \mathbb{R}^p.$$

Such a T is *minimal* if it does not contain a proper vector subspace $S \subset T$ which is also a covering matrix space.

(a) (8 points) Let T be a minimal covering matrix space and let $n = \dim T$. Prove that

$$\delta(T) \leq \binom{n}{2}.$$

(b) (2 points) Prove that for every positive integer n we can find m and p, and a minimal covering matrix space T as above such that $\dim T = n$ and $\delta(T) = \binom{n}{2}$.